MATH 2850: FIRST ORDER LINEAR EQUATIONS

DEFINITION: A **first order linear ODE** is an ODE of the form: $y' + p(x)\,y = f(x)$.

NOTE: The ODE $y' = f(x)$ is linear since here, $p(x) = 0$.

EXAMPLE: Are the following equations linear or non-linear? Explain.

- $y' - 3y = 0$

Ans: linear; $p(x) = -3$, $f(x) = 0$.

- $x\,y' + 2y = e^x$

Ans: linear; $p(x) = \dfrac{2}{x}$, $f(x) = \dfrac{e^x}{x}$.

- $y\,y' + 2y = e^x$

Ans: non-linear; cannot be written as $y' + p(x)\,y = f(x)$.

REVIEW EXERCISE: Simplify $D_x\left[y\,e^{\int p(x)\,dx}\right]$

Ans: $D_x\left[y\,e^{\int p(x)\,dx}\right] = e^{\int p(x)\,dx}\,y' + p(x)\,e^{\int p(x)\,dx}\,y$

SOLVING FIRST ORDER LINEAR ODEs

INTEGRATING FACTOR: To solve $y' + p(x)\,y = f(x)$ multiply both sides by $e^{\int p(x)\,dx}$:

$$\begin{aligned}
y' + p(x)\,y &= f(x) \\
e^{\int p(x)\,dx}\left(y' + p(x)\,y\right) &= e^{\int p(x)\,dx} f(x) \\
e^{\int p(x)\,dx} y' + p(x)\,e^{\int p(x)\,dx} y &= e^{\int p(x)\,dx} f(x) \\
D_x\left[y\,e^{\int p(x)\,dx}\right] &= e^{\int p(x)\,dx} f(x) \\
y\,e^{\int p(x)\,dx} &= \int e^{\int p(x)\,dx} f(x)\,dx \\
y &= e^{-\int p(x)\,dx}\int e^{\int p(x)\,dx} f(x)\,dx
\end{aligned}$$

The factor $\mu(x) = e^{\int p(x)\,dx}$ is called an **integrating factor** since multiplying the left hand side of the ODE by this factor results in an exact differential which we integrate to isolate the solution.

While you may choose to memorize the form of the solution in the derivation above, it is best to work through the procedure in each specific case.

STEPS TO SOLVING A FIRST ORDER LINEAR ODE:

1. Write the ODE in the form $y' + p(x)\,y = f(x)$.
2. Find the integrating factor $\mu(x) = e^{\int p(x)\,dx}$.
3. Multiply **both sides** of the ODE by $\mu(x)$.
4. Integrate both sides of the ODE.
5. Solve for y.

NOTE: We are assuming both $p(x)$ and $f(x)$ are continuous on a common open interval to guarantee the integrals are defined. As we'll see, doing so guarantees we get the **general** solution to the ODE in this manner.

EXAMPLE: Solve the following ODEs.

- $y' - 3y = 0$

Ans: $y = C\, e^{3x}$.

- $y' - 3y = x$

Ans: $y = C\, e^{3x} - \dfrac{1}{3}x - \dfrac{1}{9}$.

DEFINITION: The ODE $y' + p(x)\, y = f(x)$ is called **homogeneous** if $f(x) = 0$. That is, if $y' + p(x)\, y = 0$.

NOTE 1: $y = 0$ is a solution to $y' + p(x)\, y = 0$ for any function $p(x)$. Here, $y = 0$ is called the **trivial** solution.

Non-zero solutions are called **non-trivial**.

NOTE 2: In the above example, $y' - 3y = 0$, is called the **associated** homogeneous ODE of $y' - 3y = x$.

The solution to $y' - 3y = 0$, namely $y = C\, e^{3x}$ appears in the solution of $y' - 3y = x$: $y = C\, e^{3x} - \dfrac{1}{3}x - \dfrac{1}{9}$.

This will always happen as summarized in the following theorem:

THEOREM: The solution to a first order linear ODE $y' + p(x)\, y = f(x)$ has the form $y = y_c + y_p$ where:

- y_c is the (general) solution to the associated homogeneous ODE: $y' + p(x)\, y = 0$.

  NOTE: y_c is called the **complementary solution**.

- y_p is a solution to the original non-homogeneous ODE: $y' + p(x)\, y = f(x)$.

  NOTE: y_p is called the **particular solution**.

We'll see why this is true in the next example.

EXAMPLE: Suppose y and y_p are solutions to the ODE: $y' + p(x)\,y = f(x)$.

Prove $y_c = y - y_p$ is a solution to the associated homogeneous ODE: $y' + p(x)\,y = 0$.

EXAMPLE: Solve the IVP: $2x\,y' - x\,y = y$, $y(-4) = 2$. State the interval of validity.

Ans: $y = \sqrt{-x}\,e^{x/2+2}$ on $(-\infty, 0)$.

EXAMPLE: Solve the IVP: $(x^2+1)y' + 4xy = \dfrac{2}{x^2+1}$, $y(1) = 3$. State the interval of validity.

Ans: $y = \dfrac{2x+10}{(x^2+1)^2}$ on $(-\infty, \infty)$

EXAMPLE: Find the solutions to the following growth IVPs:

- (Unlimited Growth): $P'(t) = r\,P(t)$, $P(0) = P_0$. Assume $r, P_0 > 0$.

Ans: $P(t) = P_0\, e^{rt}$

- (Limited Growth): $P'(t) = r\,(L - P(t))$, $P(0) = P_0$. Assume $r, P_0 > 0$ and $0 < P(t) < L$.

Ans: $P(t) = L + (P_0 - L)\, e^{-rt}$

EXAMPLE: Solve the IVP: $y' = 2xy + 1,\ y(2) = 4$. State the interval of validity.

Ans: $y = e^{x^2} \int_2^x e^{-t^2}\, dt + 4e^{x^2-4}$

THEOREM: EUT (Existence and Uniqueness Theorem) for First Order Linear ODEs:

If p and f are continuous in an open interval containing x_0, then the IVP:

$$y' + p(x)\,y = f(x), \qquad y(x_0) = y_0$$

has a unique solution. Moreover, the solution has the form:

$$y = y_1(x) \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt + \frac{y_1(x)}{y_1(x_0)}\,y_0$$

where $y_1(x)$ is a (non-trivial) solution to the associated homogeneous equation: $y' + p(x)\,y = 0$.

We break the proof into steps:

STEP 1: Solve the associated homogeneous problem: $y' + p(x)\,y = 0$, $y(x_0) = y_0$ using an integrating factor.

STEP 2: Show if y_1 and y_2 are solutions to the associated homogeneous equation: $y' + p(x)\,y = 0$, then there is a constant c so that $y_1 = c\,y_2$.

STEP 3 We'll show only solution to the IVP $y' + p(x)\,y = 0$, $y(x_0) = 0$ is $y = 0$. (The zero function.)

STEP 4: Show if y_1 and y_2 are two solutions to $y' + p(x)\,y = f(x)$, $y(x_0) = y_0$, then $y_3 = y_1 - y_2$ is a solution to $y' + p(x)\,y = 0$, $y(x_0) = 0$. From STEP 3, we'll know that $y_3 = 0$ so $y_1 = y_2$.

STEP 5: Solve $y' + p(x)\,y = f(x)$, $y(x_0) = y_0$ using an integrating factor to show a solution exists. We'll then know from STEP 4 that this solution is unique.

HOMEWORK: pg. 41: 1-45 odd